Isospectral membranes: a connection between shape and density

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Abstract

A connection is established via conformal maps between vibrating membranes that are isospectral with respect to shape and those that are isospectral with respect to density. In particular, inhomogeneous circular membranes are constructed that are isospectral to polygonal membranes of uniform density via the Schwarz–Christoffel mapping. Although some corners of the polygons lead to singularities in the constructed densities, the densities are shown to be integrable.

PACS numbers: 02.30.Fn, 02.30.Jr, 02.60.Lj
Mathematics Subject Classification: 30C30, 30E25, 35J55, 35P99, 35R30, 74K15

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The vibrating membrane has many applications including drums, microphones, loudspeakers [16] and transducers [17]. A variety of techniques for the solution of the associated boundary value problem have been studied [14]. We use one of these techniques, conformal mappings, here to investigate the connection between two classes of isospectral membranes, i.e. membranes whose eigenfrequency spectra are identical. Conformal mappings have often been used to solve boundary value problems arising in applications. The basic premise is that the transformed problem is posed over a nicer geometry, thereby making the problem more tractable. This arises in the study of vibrating membranes [14], electromagnetism [2] and fluid flow [15]. Although conformal mappings are generally used only to change the geometry of a problem, they can also be used to change the coefficients of a problem. Here we use a
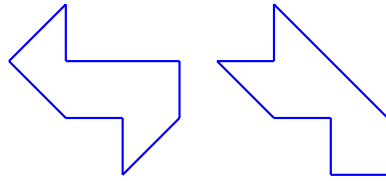
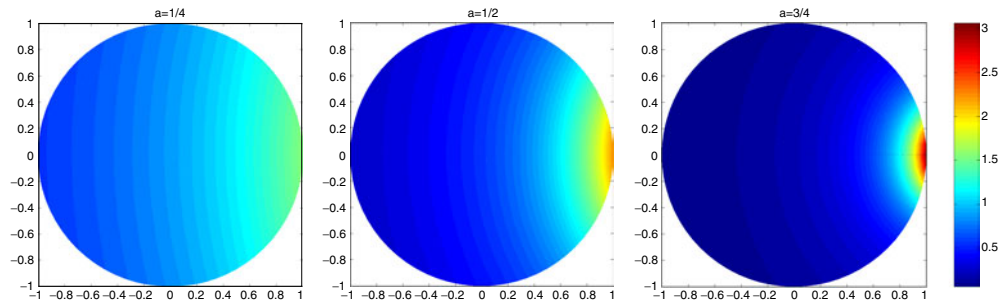**Figure 1.** Isospectral shapes found by Gordon, Webb and Wolpert.

Figure 2. Densities isospectral to a uniform circular membrane found by Gottlieb for $a = 1/4, 1/2, 3/4$.

conformal map to study the connection between membranes that are isospectral with respect to shape and those that are isospectral with respect to density.

In the context of the homogeneous vibrating membrane, the existence of isospectral shapes allows us to answer 'no' to the famous question 'Can one hear the shape of a drum?' [12]. Such shapes were first constructed for the two-dimensional membrane by Gordon, Webb and Wolpert [4, 3, 9] using a result of Sunada [22]. The examples constructed included the polygons shown in figure 1.

Gottlieb [10] used a conformal map to construct a class of inhomogeneous unit circular membranes with density $\rho(x, y)$ that were isospectral to a homogeneous unit circular membrane with constant density ρ_0:

$$\rho(x, y) = \frac{\rho_0(1 - a^2)^2}{[(1 - ax)^2 + a^2 y^2]^2}$$

for each $a \in (0, 1)$. Figure 2 shows examples of the Gottlieb isospectral densities. Recently, Gottlieb [11] generalized his previous result and established a similar construction showing the isospectrality of two inhomogeneous unit circular membranes with densities $\rho_1(x, y)$ and $\rho_2(x, y)$ respectively:

$$\rho_2(x, y) = \frac{\rho_1(\bar{x}, \bar{y})(1 - a^2)^2}{[(1 - ax)^2 + a^2 y^2]^2}$$

for each $a \in (0, 1)$, with

$$\bar{x} = \frac{(x - a)(1 - ax) - ay^2}{(1 - ax)^2 + a^2 y^2}, \qquad \bar{y} = \frac{(1 - a^2)y}{(1 - ax)^2 + a^2 y^2}.$$

2. Conformal mappings of inhomogeneous membranes

The technique used is as follows. Let Ω and $\mathcal{O}$ be domains in the complex planes $z = x + iy$ and $w = u + iv$. Let $w = F(z)$ be a one-to-one analytic mapping from Ω to $\mathcal{O}$. For $\mathcal{O}$ and Ω strict simply-connected subsets of the complex plane, the existence of the mapping F follows by the Riemann mapping theorem (see e.g. [19, theorem 14.8]). The inhomogeneous clamped membrane with areal density distribution $\rho_1(x, y)$

$$-\Delta U = \lambda \rho_1(z) U \quad \text{in } \Omega \qquad U = 0 \quad \text{on } \partial\Omega$$

is mapped to the inhomogeneous clamped membrane with areal density distribution $\rho_2(u, v)$

$$-\Delta V = \lambda \rho_2(w) V \quad \text{in } \mathcal{O} \qquad V = 0 \quad \text{on } \partial\mathcal{O}$$

with

$$\rho_1(z) = \rho_2(w)|F'(z)|^2,$$

cf Saff and Snider [20].

Since we know that there are isospectral polygonal membranes with uniform density and isospectral circular membranes with varying densities, we will use a conformal map between polygons and circles to construct a new class of isospectral densities for circular membranes that are also isospectral to polygonal membranes.

2.1. Schwarz–Christoffel mapping

The Schwarz–Christoffel mapping is the conformal mapping $w = F(z)$ that maps the unit disc $\mathcal{D} = \{z : |z| < 1\}$ to a polygon $\mathcal{P}$. If the polygon has n vertices w_k, given in counter-clockwise order, and interior angles $\alpha_k \pi$ $(k = 1, \ldots, n)$, then the map has the form

$$F(z) = A \int^z \prod_{k=1}^n \left(1 - \frac{z}{z_k}\right)^{-\beta_k} dz + B \tag{1}$$

where A and B are complex constants, $\beta_k = 1 - \alpha_k$ and z_k are points on the unit circle known as *prevertices* satisfying $w_k = F(z_k)$, cf Ahlfors [1], Driscoll and Trefethen [8]. For a bounded polygon with no cusps or slits, $\alpha_k \in (0, 2)$. A cusp at the vertex w_k yields $\alpha_k = 0$, while a slit yields $\alpha_k = 2$. Vertices at infinity yield $\alpha_k \in [-2, 0]$. In order for the polygon to be closed, $\sum \beta_k = 2$ and hence $\sum \alpha_k = n - 2$, where n is the number of vertices. For polygons with $n > 3$ the prevertices z_k cannot generally be computed explicitly. Their computation is the subject of a great deal of literature, notably [23, 21, 6, 24].

2.2. Density of a circular membrane resulting from a transformed polygon

Consider a homogeneous membrane on the polygon $\mathcal{P}$ with constant density, ρ_0. This polygon can be mapped to the unit disc $\mathcal{D}$ using the inverse of the Schwarz–Christoffel mapping. The resulting circular membrane is inhomogeneous and has density ρ, with

$$\rho(z) = \rho_0 |F'(z)|^2 = \rho_0 |A|^2 \prod_{k=1}^n \left|1 - \frac{z}{z_k}\right|^{-2\beta_k}.$$

Clearly, if $\beta_k > 0$ then $\rho(z)$ has singularities on the boundary of the circular membrane. In that case, $\rho(z)$ is bounded except at $z = z_k$ for $k = 1, \ldots, n$, and so

$$\int_{\mathcal{D}} \rho(z) \, dz < \infty \quad \Longleftrightarrow \quad \int_{B_\varepsilon(z_k)} \rho(z) \, dA < \infty \qquad \text{for} \quad k = 1, \ldots, n.$$

Since there are constants C_1, C_2 such that

$$C_1|z - z_k|^{-2\beta_k} \leqslant \rho(z) \leqslant C_2|z - z_k|^{-2\beta_k}$$

on the ball $B_\varepsilon(z_k)$, it suffices to consider

$$
\begin{aligned}
\int_{B_\varepsilon(z_k)} |z - z_k|^{-2\beta_k} \, \mathrm{d}A &= \int_{B_\varepsilon(z_k)} \frac{1}{|z - z_k|^{2\beta_k}} \, \mathrm{d}A \\
&= \int_{B_\varepsilon(0)} \frac{1}{|z|^{2\beta_k}} \, \mathrm{d}A \\
&= \int_0^{2\pi} \int_0^\varepsilon \frac{1}{r^{2\beta_k}} r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= 2\pi \int_0^\varepsilon r^{1-2\beta_k} \, \mathrm{d}r \\
&= \frac{\pi r^{2(1-\beta_k)}}{1 - \beta_k} \bigg|_0^\varepsilon = \frac{\pi r^{2\alpha_k}}{\alpha_k} \bigg|_0^\varepsilon .
\end{aligned}
$$

Hence

$$\int_{\mathcal{D}} \rho(z) \, \mathrm{d}A < \infty \quad \Longleftrightarrow \quad \alpha_k > 0 \qquad \text{for} \quad k = 1, \dots, n.$$

In geometric terms, this means that the unit disc density associated with a particular polygon will be integrable provided that the interior angles of the polygon $\alpha_k \pi$ are positive. Thus a polygon may have no cusps and must be bounded in order to guarantee integrability of the density of the constructed circular membrane.

3. Examples

3.1. Regular polygons mapped to circle

Let Ω be the regular N-sided polygon that inscribes the circle of radius 1. In this case, the vertices are $w_k = \exp\left(\mathrm{i}\frac{2k\pi}{N}\right)$ and the interior angles are $2\pi/N$ yielding $\alpha_k = 2/N$. Fixing the prevertices so that $z_k = w_k$, the unit disc $\mathcal{D}$ is mapped to the N-sided polygon Ω by the Schwarz–Christoffel mapping:

$$w = \int_0^z \frac{\mathrm{d}s}{(1 - s^N)^{2N}},$$

cf Kober [13]. The density of the inhomogeneous membrane associated with the disc is

$$\rho(z) = \rho_0 \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2 = \rho_0 |1 - z^N|^{-4N}.$$

Figure 3 shows the homogeneous polygon inscribed in a unit circle and its corresponding inhomogeneous density for the case $N = 8$. This density has eight singularities—one at each of the prevertices z_k corresponding to each vertex of the polygon. Since $\beta_k = 1 - \alpha_k = (N - 2)/N < 1$, the constructed density is in L^1. This means that the eigenfunctions are in L^2, [25]. Thus there is a circular inhomogeneous membrane that is isospectral to a regular polygon. Cureton and Kuttler [5] and Molinari [18] use this circular inhomogeneous membrane to compute the eigenvalues of the Laplacian over regular polygons for vastly different applications.
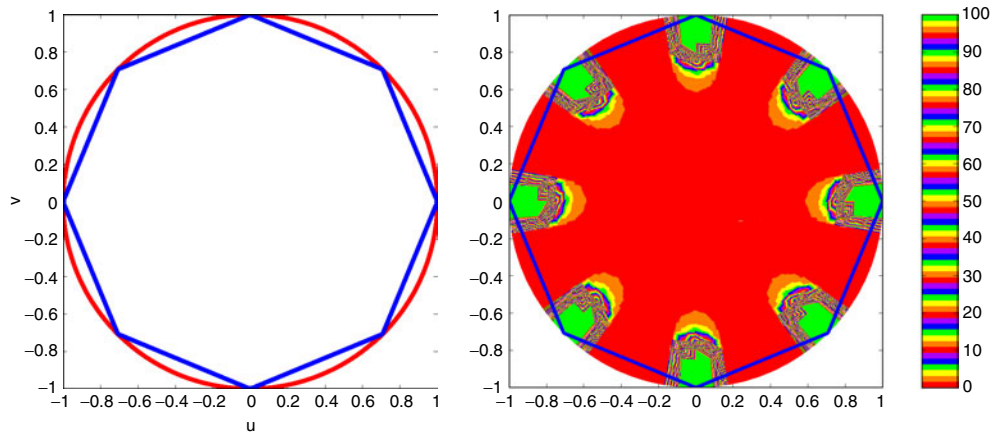
Figure 3. $N = 8$, an eight-sided homogeneous polygon inscribed in a unit circle and its corresponding inhomogeneous density.
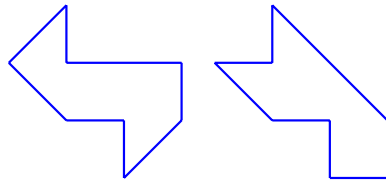

Figure 4. Ω_1 *bilby* and Ω_2 *hawk*.

3.2. Gordon, Webb and Wolpert's shapes mapped to circle

Let Ω_1 and Ω_2, shown in figure 4, be the isospectral shapes discovered by Gordon, Webb and Wolpert [4, 3, 9]. These particular shapes were constructed using a right isosceles triangle and a series of reflections prescribed by two equivalent Cayley graphs [9]. Other triangles can be used to form similar pairs of isospectral shapes. These particular shapes were nicknamed *bilby* and *hawk* in [7].

The isospectrality of these shapes means that for homogeneous membranes with constant density ρ_0, the boundary value problems

$$-\Delta u = \lambda \rho_0 u \quad \text{in } \Omega_i \qquad u = 0 \quad \text{on } \partial\Omega_i$$

have the same eigenvalues for Ω_1 and Ω_2. Using Schwarz–Christoffel mappings, we transform the unit discs $\mathcal{D}_1$ and $\mathcal{D}_2$ to Ω_1 and Ω_2 respectively. Prevertices for these maps were computed using the SC toolbox [8], and are shown for Ω_1 and Ω_2 in figure 5. Although the prevertices appear to cluster, they are in fact distinct because F is one-to-one, $w_k = F(z_k)$ and the polygon vertices w_k are distinct.

Consider the inhomogeneous membrane associated with each disc. Each has a density of the form

$$\rho(z) = \rho_0 |F'(z)|^2 = \rho_0 |A|^2 \prod_{k=1}^{n} \left| 1 - \frac{z}{z_k} \right|^{-2\beta_k}.$$

The values of α_k and β_k for each of the polygons Ω_1 and Ω_2 are given in table 1. These densities each have six singularities—one at each of the prevertices z_k corresponding to a vertex whose interior angle $\alpha_k \pi$ is less than π. All of the β_k satisfy $\beta_k < 1$ and so each of the
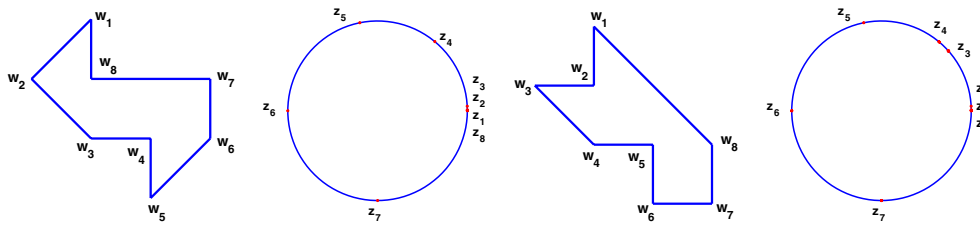
Figure 5. Ω_1, $\mathcal{D}_1$, Ω_2 and $\mathcal{D}_2$, showing vertices and prevertices.
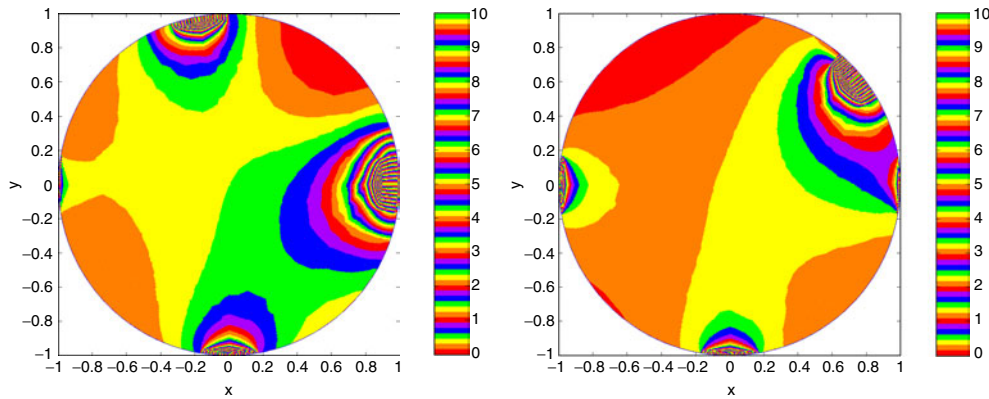
Figure 6. Gordon, Webb and Wolpert density for $\mathcal{D}_1$ and $\mathcal{D}_2$.

Table 1. Angle information.

Ω_1	α_k	1/4	1/2	3/4	3/2	1/4	3/4	1/2	3/2
	β_k	3/4	1/2	1/4	−1/2	3/4	1/4	1/2	−1/2
Ω_1	α_k	1/4	3/2	1/4	3/4	3/2	1/2	1/2	3/4
	β_k	3/4	−1/2	3/4	1/4	−1/2	1/2	1/2	1/4

constructed densities is in L^1 with eigenfunctions in L^2. Thus there are physically meaningful densities for circular inhomogeneous membranes that are isospectral to each other and to the homogeneous shapes constructed by Gordon, Webb and Wolpert. These densities are shown in figure 6.

Isospectrality of the *bilby* and *hawk* shapes also holds for Neumann boundary conditions [4, 3] and has been verified numerically by Driscoll and Gottlieb [7]. Analysis identical to that used above shows that, in the case of Neumann boundary conditions, inhomogeneous circular membranes can be constructed that are isospectral to homogeneous membranes with the bilby and hawk shapes.

Acknowledgments

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